FRACTIONAL STRAIN-GRADIENT PLASTICITY

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ABSTRACT. We develop a strain-gradient plasticity theory based on fractional derivatives of plastic strain and assess its ability to reproduce the scaling laws and size effects uncovered by the recent experiments of Mu et al. (2014, 2016, 2017) on copper thin layers undergoing plastically constrained simple shear. We show that the size-scaling discrepancy between conventional strain-gradient plasticity and the experimental data is resolved if the inhomogeneity of the plastic strain distribution is quantified by means of fractional derivatives of plastic strain. In particular, the theory predicts that the size scaling exponent is equal to the fractional order of the plastic-strain derivatives, which establishes a direct connection between the size scaling of the yield stress and fractionality.

1. INTRODUCTION

This paper is concerned with the development of a strain-gradient plasticity model based on fractional derivatives and with the assessment of its ability to reproduce the scaling laws and size effects uncovered by the recent experiments of Mu et al. [1, 2, 3] on copper thin films under constrained simple shear. The work is motivated by a discussion session led by J. W. Hutchinson at a recent IUTAM symposium¹, where he pointed out what appears to be an essential discrepancy between conventional strain-gradient plasticity (SGP) predictions and experimental observations [1, 2, 3], and identified that discrepancy as an important and unresolved challenge. Hutchinson specifically exemplified the gap between theory and experiment by means of the “canonical problem posed and analyzed in most papers on SGP formulations” [2], namely, constrained shear of thin layers. In this paper, we show that the scaling discrepancy in the shear problem [1, 2, 3] can be resolved if the effects of a non-homogenous plastic strain field are accounted for by means of fractional derivatives of plastic strain.

There is at present ample experimental evidence that plastic deformation is scale dependent. Generally, as some measure of size related to a plastically deforming material decreases, the strength increases. For instance, based on experiments on steel Hall [4] and Petch [5] derived their classical scaling law according to which the yield stress of polycrystalline metals is inversely proportional to the square root of the grain size. Ashby [6] pointed

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out that constraints on a crystal lattice during plastic flow introduce so-called geometrically necessary dislocations that contribute to the flow stress through a Taylor-type hardening relation \[7\]. The implication is that, when gradients of plastic deformation are imposed by the microstructure due to secondary phases, disperse particles, grain boundaries, or by an external deformation field, plastic deformation proceeds at an elevated stress relative to what could be expected from a uniaxial stress-strain test on a well-annealed specimen.

Since the work of Hall and Petch, a large number of experiments have been reported that reveal a plastic size dependence. In nano- and micro-indentation, the results have been found to be strongly dependent on indentation depth by, e.g., Ma and Clarke [8] on single crystals of silver, and Poole, Ashby and Fleck [9] on copper polycrystals, where the hardness scales inversely proportionally to the indentation depth raised to the power \(1/2\). Fleck et al. [10] measured torque versus twist for thin copper wires of different diameters. The results show a clear strengthening effect as the wire diameter is decreased, with the strengthening accelerating at smaller diameters. Stölken and Evans [11] and later Ehrler et al. [12] conducted small-scale bending experiments on thin nickel foils where the prescribed through-thickness gradient gives rise to a distinct strengthening. Dunstan and Bushby [13] compiled data from a large set of various small-scale experiments and found exponents in the range of 0.2 to 1 when fitting a Hall-Petch type equation to the inverse grain size.

Recently, a new experimental protocol has been developed by Mu et al. [1, 2] consisting of a thin layer of ductile copper sandwiched between stiff and brittle ceramic coatings (CrN and Si). Simple shear conditions can be induced in the metal by means of micro-pillar compression of the layered ceramic/metal/ceramic with the layers oriented at 45 degrees to the load axis. Since the ceramic materials do not deform plastically, the interface between metal and ceramic is assumed to constrain plastic slip. Therefore, a plastic strain gradient necessarily develops across the thickness of the copper layer. As the thickness of the copper layer is decreased, the shear stress required for large-scale plastic flow is increased. When fitting the yield stress in shear to a power law function of the inverse layer thickness, Mu et al. [1, 2] found exponents of the order of 0.2 in the as-deposited state and 0.7 when annealed. Compression tests were also performed on the material system and the strength shows a power law dependence on inverse thickness with exponent 1.0 in the as-deposited state and 0.7 in the annealed state. Figure 1 shows selected experimental measurements together with fitted trend lines and a strain-gradient plasticity model output. The same experimental protocol has also been used to investigate thin layers of Ti and Cr [3], but with less conclusive results due to a larger scatter and fewer layer thicknesses.
Figure 1. Shear flow stress as a function of thickness for Cu layers. A discrepancy between power law fits and SGP model output are indicated by solid and dashed lines, respectively. Insert shows SEM image of experimental setup. From Mu et al. [2], figure 4(a), courtesy of Cambridge University Press.

In parallel to the increasing experimental evidence of size-dependent plastic deformation, a large effort has been devoted to developing continuum theories that account for the observed phenomena, starting with the pioneering work of Aifantis [14], Fleck and Hutchinson [10, 15] and Nix and Gao [16]. A coarse, but relevant, classification divides the existing theories into standard or lower order and higher-order theories. The standard-order theories incorporate some measure of the plastic strain gradient into the hardening relation but leave the structure of the governing equations and boundary conditions unchanged. This type of theory cannot predict a size dependence of the yield stress but can exhibit a size-dependent increase in work hardening. By contrast, higher-order theories incorporate the plastic strain gradient into a variational principle for plastically-deforming bodies. This type of theory changes the structure of the governing equilibrium equations by introducing a couple-stress tensor and higher-order boundary conditions. Higher-order isotropic strain-gradient plasticity models have the ability to predict qualitatively both the size-dependent increase in yield stress and the increase in work hardening. However, they often fail to be quantitatively predictive outside the range of experiments they are fitted to. For instance, as noted by Evans and Hutchinson [17] the size-dependent increase of the
bending moment at yielding, and subsequent hardening, is too large to accommodate the experimental data [12] when fitting to a single constant constitutive length scale.

Gradient plasticity models predict a yield stress of the form

\[ \sigma_y = \sigma_0 \left[ 1 + \left( \frac{\ell}{h} \right)^\alpha \right], \]

where \( \ell \) is a constant constitutive length scale, \( h \) is an appropriate size measure of the plastically deforming region and \( \alpha \) is a scaling exponent. Higher-order formulations of strain-gradient plasticity invariably predict an exponent \( \alpha = 1 \) [15, 18, 19, 20, 21, 22, 23, 24], i.e., they predict that the yield stress is in inverse proportion to size. Thus, this inverse-size proportionality scaling appears to be an essential attribute of conventional strain-gradient plasticity arising directly from its differential structure. However, as already noted the available experimental data suggests exponents within the extreme values of \( \alpha = 0 \), corresponding to local plasticity, and \( \alpha = 1 \), corresponding to conventional strain-gradient plasticity.

Within the context of higher-order strain-gradient plasticity, there is a large number of possible extensions of conventional strain-gradient plasticity that could potentially alleviate the scaling discrepancy. A case in point consists of defining ad hoc internal variables combining contributions from plastic strain and plastic-strain gradients. Most commonly, root-mean-square combinations have been considered [25, 17], though other expressions have been used as well [19, 26, 27, 23]. Except for some changes to the scaling at \( \ell/h \ll 1 \), i.e., close to the large scale limit, there is no indication that this procedure produces size scalings with \( \alpha < 1 \). Another modeling strategy [28, 17, 29] modifies the constitutive length scale \( \ell \) as a function of the accumulated of plastic deformation. While this extension can significantly improve the fit to some experimental results, it does not alter the size scaling of the underlying theory.

We therefore surmise that a proper accounting of the experimentally observed size scaling requires relaxing the differential structure of strain-gradient plasticity, which is overly stiff. We accomplish this relaxation by allowing for a dependence of the free energy on fractional derivatives of the plastic strain. We refer to the resulting model as fractional strain-gradient plasticity (FSGP). We recall that fractional derivatives are widely used in mechanics, e.g., in the formulation of models of viscoelasticity [30]. Fractional derivatives have also been introduced in deformation theories of inelasticity in connection with models of ductile fracture [31, 32, 33]. The aim of the present work is to formulate a practical flow-theory of fractional strain-gradient plasticity and demonstrate its ability to match the experimentally-observed range of size-scaling exponents in constrained layers undergoing simple shear [1, 2, 3]. Since the definition and computation of fractional derivatives in multiple dimensions is quite technical (cf., e.g., [34, 35, 36]), we choose to incorporate the fractional differential structure...
directly into the total free energy of the body by means of equivalent double-integral expressions [37]. The resulting free energy is thus explicitly defined, non-local on account of its double-integral representation, and its fractional differential character is set simply by an appropriate choice of exponents in the interaction kernel. We show that, when applied in the shear layer configuration, the theory predicts scaling relations of the form (1) with \( \alpha \) equal to the fractional order of plastic strain-gradient differentiation. In particular, the observed experimental scaling can be exactly matched by an appropriate choice of fractional differential order.

The paper is organized as follows. In section 2, a variational formulation of strain-gradient plasticity is presented and specialized to deformation-theory. Within this framework, the problem of simple shear of a plastically constrained layer is solved in section 3 and the resulting size scaling is discussed. In section 4 a fractional strain-gradient plasticity theory is presented and the resulting size scaling is derived from the solution to the simple shear problem in section 4.1. The paper is concluded in section 5.

2. Strain-gradient plasticity

The theoretical basis of strain-gradient plasticity is well-known but may stand a brief review in the interest of completeness and to set notation. We specifically follow the formulation of Gudmundson [18] and Fleck and Willis [20, 21]. The notation adopted in the sequel is adapted from the paper of Gudmundson.

2.1. Constitutive framework. We assume throughout linearized kinematics and postulate the conventional additive decomposition of strain

\[ \varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p}, \]

into elastic and plastic components, \( \varepsilon_{ij}^{e} \) and \( \varepsilon_{ij}^{p} \), respectively. For metals, the plastic deformations are nearly incompressible,

\[ \varepsilon_{kk}^{p} = 0, \]

over a broad range of pressures.

We additionally assume a free energy density of the form

\[ \psi = \psi(\varepsilon^{e}, \varepsilon^{p}, D\varepsilon^{p}), \]

where we write \( \varepsilon^{e} = \varepsilon_{ij}^{e} \), \( \varepsilon^{p} = \varepsilon_{ij}^{p} \) and \( D\varepsilon^{p} = \varepsilon_{ij,k}^{p} \). We note that the free energy is assumed to depend on the local total and plastic strains, but also on the plastic strain gradients. This latter dependence renders the solid non-local and global averages may be expected to deviate from volume scaling in general. A common choice of the free energy is

\[ \psi(\varepsilon^{e}, \varepsilon^{p}, D\varepsilon^{p}) = \psi_{e}(\varepsilon^{e}) + \psi_{p}(\varepsilon^{p}) + \psi_{g}(D\varepsilon^{p}), \]

with Hookean elastic response

\[ \psi_{e}(\varepsilon^{e}) = \frac{1}{2} C_{ijkl} \varepsilon_{ij}^{e} \varepsilon_{kl}^{e}. \]
The corresponding equilibrium stress-strain relation is

\[ \sigma_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}} (\varepsilon^e, \varepsilon^p, D\varepsilon^p) = C_{ijkl} \varepsilon_{kl}, \]

where \( \sigma_{ij} \) is the Cauchy stress. A common assumption for the local part of the inelastic free energy is

\[ \psi_p = A \varepsilon_0^{-m} \left( \varepsilon_{ij} \varepsilon_{ij} \right)^{m+1}, \]

corresponding to power-law hardening, where \( A \) is a hardening modulus, \( \varepsilon_0 \) is a reference strain and \( m \) is a hardening exponent. From experimental data on metals [38] we expect \( 0 \leq m < 1 \). A general form for the gradient part of the free energy is [39]

\[ \psi_g = B \varepsilon_0^{-n} \left( \varepsilon_{ij,k} \varepsilon_{ij,k} \right)^{n+1}, \]

where \( B \) is a hardening modulus, \( \ell \) is a constitutive length scale parameter and \( n \) is a non-local hardening exponent.

The evolution of the plastic strain is further constrained by a dissipation potential \( g(\dot{\varepsilon}^p, D\dot{\varepsilon}^p) \) in the manner to be made precise subsequently. A common choice of the dissipation potential is

\[ g(\dot{\varepsilon}^p, D\dot{\varepsilon}^p) = \sigma_0 \left( \varepsilon_{ij} \varepsilon_{ij} \right)^{\frac{m+1}{2}}, \]

which reduces to rate-independent Mises plasticity in the local limit \( \ell = 0 \).

2.2. Variational formulation of boundary-value problems. The preceding elements of constitutive theory need to be combined with the equations of equilibrium and compatibility and with displacement and traction boundary conditions in order to define the general static equilibrium initial-boundary-value problem.

Variational principles supply a compact means of formulating said problems. Thus, the stable equilibrium configurations of the body are characterized by the principle of minimum potential energy

\[ E(u) = \int_\Omega \left( \psi(\varepsilon(u)) - \varepsilon^p, \varepsilon^p, D\varepsilon^p \right) \, dV - \int_{\Gamma_N} t_i u_i \, dS \rightarrow \min, \]

where \( u_i(x, t) \) is the displacement field over the domain \( \Omega \) of the body, subject to boundary conditions

\[ u(x, t) = \bar{u}_i(x, t) \]

on the Dirichlet or displacement boundary \( \Gamma_D = \Gamma \setminus \Gamma_N \). In (11), \( \varepsilon_{ij}(u) \) denotes the strain operator,

\[ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \]

\( f_i(x, t) \) represents a general distribution of body forces and \( t_i(x, t) \) a general distribution of applied tractions over the Neumann or traction boundary \( \Gamma_N \).
In addition, the rate of plastic straining is characterized by the minimum rate problem

$$G(\dot{\varepsilon}^p) = \int_{\Omega} \left(g(\dot{\varepsilon}^p, D\dot{\varepsilon}^p) - \sigma_{ij}\dot{\varepsilon}_{ij}^p\right) dV \to \min.$$  \hfill (14)

The elastic equilibrium and rate problems jointly define an evolution problem for the state of hardening and plastic deformation of the solid.

2.3. Rigid-plastic and deformation-theory approximations. In subsequent developments, we resort to a number of approximations in order to render analysis tractable. When elastic strain rates are negligible, e. g., at plastic collapse, we have

$$\varepsilon_{ij}^p \approx \varepsilon_{ij}(\dot{u}).$$  \hfill (15)

Under these conditions, the free energy (11) reduces to

$$E(u) = \int_{\Omega} \left(\psi_p(\varepsilon(u)) + \psi_g(D\varepsilon(u)) - f_iu_i\right) dV - \int_{\Gamma_N} t_iu_i dS,$$

where we have assumed (5) for definiteness. The rate problem (14) correspondingly reduces to

$$G(\dot{u}) = \int_{\Omega} \left(g(\dot{\varepsilon}(\dot{u}), D\dot{\varepsilon}(\dot{u})) + \frac{d}{dt}E(u)\right) dV \to \min,$$

which directly characterizes the evolution of the displacement field. If, in addition, the body undergoes proportional loading and the dissipation function $g(\dot{\varepsilon}^p, D\dot{\varepsilon}^p)$ is homogeneous of degree one, as in (10), the deformation is likewise proportional in time and the minimum problem (17) reduces to

$$F(u) = \int_{\Omega} \left(g(\varepsilon(u), D\varepsilon(u)) + \psi_p(\varepsilon(u)) + \psi_g(D\varepsilon(u)) - f_iu_i\right) dV$$

$$- \int_{\Gamma_N} t_iu_i dS \to \min,$$  \hfill (18)

for any time $t$. We note that the minimum problem (18) has the structure typical of energy minimization. However, the effective of deformation-theoretical energy is a combination of free energy and dissipation.

3. Simple shear of plastically constrained layer

We wish to make contact with the experiments of Mu et al. [3] as a means of assessing the ability of deformation-gradient plasticity to describe the behavior of actual materials. The tests consist of compressed micro-pillar composite specimens in which polycrystalline Cu thin films are sandwiched between CrN and Si. The main focus of the analysis is to establish the scaling relations that govern the dependence of the flow stress on the thickness of the Cu film.

To a first approximation, the deformation of the Cu films can be idealized as uniform simple shear in an unbounded layer of thickness $2h$, with
plastic flow constraint at the interfaces. We therefore consider a layer of Cu of infinite extension in the $x$ and $z$-directions and thickness $2h$ in the $y$-direction, schematically illustrated in Figure 2(a). Under these conditions, the only non-zero displacement component is $u_x \equiv u(y)$ and the only non-zero component of strain is $\gamma = 2\varepsilon_{xy} = u_{,y}(y) \equiv \gamma(y)$. In addition, for a given average shear strain $\bar{\gamma} \geq 0$ the deformation-theory minimum problem (18) reduces to

\[
\text{(19a) Minimize: } F(\gamma) = \int_{-h}^{+h} \left( \psi_p(\gamma(y)) + \psi_g(\gamma_{,y}(y)) \right) \, dy, \\
\text{(19b) subject to: } \frac{1}{2h} \int_{-h}^{+h} \gamma \, dy = \bar{\gamma}, \quad \text{and} \quad \gamma(-h) = \gamma(h) = 0,
\]

where we assume local dissipation $g(\dot{\gamma})$ and we lump it together with $\psi_p(\gamma)$ for economy of notation. We specifically consider hardening of the power-law type,

\[
\psi_p(\gamma) = \frac{A}{m+1} |\gamma|^m + \tau_0 |\gamma|, \quad \psi_g(\gamma_{,y}) = \frac{B}{n+1} |\gamma_{,y}|^{n+1}.
\]

### 3.1. Ideal plasticity and linear-growth nonlocal energy.

The experiments of Mu et al. [3] indeed show hardening saturation after modest amounts of plastic strain, which suggests the choice $m = 0$ and $A = 0$. If, in addition, we assume an nonlocal energy with linear growth, $n = 0$, problem (20) reduces to

\[
\text{(21a) Minimize: } F(\gamma) = \int_{-h}^{+h} \left( \tau_0 |\gamma(y)| + \tau_0 \ell |\gamma_{,y}(y)| \right) \, dy, \\
\text{(21b) subject to: } \frac{1}{2h} \int_{-h}^{+h} \gamma \, dy = \bar{\gamma}, \quad \text{and} \quad \gamma(-h) = \gamma(h) = 0,
\]

where we have set $B = \tau_0 \gamma_0$. 

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Figure 2. (a) Simple shear of a layer of thickness $2h$ sandwiched between rigid supports. (b) Illustration of constrained shear strain profile, $\gamma(y)$, with average shear strain, $\bar{\gamma}$, indicated by dashed line.
Nonlocal energies with linear growth are distinguished in that they allow for jumps in the plastic strain. We therefore consider a test strain distribution jumping discontinuously from zero at \( y = \pm h \) to a constant strain \( \gamma \) in the interior of the layer, \(-h < y < h\). The corresponding minimum problem follows from (21) as
\[
F = 2h \tau_0 |\gamma| + 2\ell \tau_0 |\gamma| - 2h \bar{\tau} \gamma \rightarrow \text{min!}
\]
where we have introduced a Lagrange multiplier \( \bar{\tau} \), the average shear stress across the layer, in order to enforce the average constraint (21b). Evidently, if
\[
|\bar{\tau}| < \frac{h + \ell}{h} \tau_0,
\]
the only minimizer of (22) is \( \bar{\gamma} = 0 \). Conversely, for non-zero minimizers to be possible, we must have
\[
|\bar{\tau}| = \frac{h + \ell}{h} \tau_0,
\]
which introduces a yield condition with effective shear yield stress
\[
\tau_\ell = \tau_0 \left[ 1 + \frac{\ell}{h} \right].
\]
As expected, the effective shear yield stress depends on the thickness of the layer. For thin layers, \( h \ll \ell \), from (25) we have the scaling
\[
\frac{\tau_\ell}{\tau_0} \sim \left( \frac{\ell}{h} \right)^{1}.
\]
Thus the theory predicts that the effective shear yield stress scales in inverse proportion to the thickness of the layer.

3.2. Effect of plastic strain hardening. Next, we consider the effect of plastic strain hardening, \( m > 0 \) and \( A > 0 \). In this range, problem (20) reduces to
\[
\text{Min: } F(\gamma) = \int_{-h}^{+h} \left( \frac{A}{m + 1} \left[ \frac{\gamma(y)}{\gamma_0} \right]^{m+1} + \tau_0 |\gamma(y)| + \tau_0 \ell |\gamma_{,y}(y)| \right) dy,
\]
subject to:
\[
\frac{1}{2h} \int_{-h}^{+h} \gamma dy = \bar{\gamma}, \quad \text{and} \quad \gamma(-h) = \gamma(h) = 0.
\]
Since the nonlocal energy remains of linear growth, we again consider a test strain distribution jumping discontinuously from zero at \( y = \pm h \) to a constant strain \( \gamma \) in the interior of the layer, \(-h < y < h\). The corresponding minimum problem now follows from (27) as
\[
F = 2h \left( \tau_0 |\gamma| + \frac{A}{m + 1} \left[ \frac{\gamma}{\gamma_0} \right]^{m+1} + 2\ell \tau_0 |\gamma| - 2h \bar{\tau} \gamma \rightarrow \text{min!}
\]

As in the ideally-plastic case, in the rate (23) the only minimizer of (28) is \( \bar{\gamma} = 0 \) and, conversely, for non-zero minimizers to be possible, we must have

\[
|\bar{\tau}| \geq \frac{h + \ell}{h} \tau_0.
\]

which again sets forth a yield condition with effective shear yield stress (25). In particular, the scaling relation (26) of the initial shear yield stress \( \tau_\ell \) persists in the presence of hardening. In the range (29), the Euler-Lagrange equation of (28) is

\[
\frac{\partial F}{\partial \bar{\gamma}} = 2h \left( \tau_0 + H \frac{\bar{\gamma}}{\gamma_0} |m\right) \text{sign}(\bar{\gamma}) + 2\ell \tau_0 \text{sign}(\bar{\gamma}) - 2h \bar{\tau} = 0,
\]

where \( H = A/\gamma_0 \) is a hardening modulus. The average stress-strain relation is now obtained by solving (30) for \( \bar{\tau} \), with the result

\[
\bar{\tau} = \left( \tau_\ell + H \frac{\bar{\gamma}}{\gamma_0} |m\right) \text{sign}(\bar{\gamma}).
\]

We see from this relation that, under the assumptions of the analysis, only the initial yield is affected by nonlocal hardening and the subsequent strain hardening remains unaffected.

3.3. Effect of nonlocal hardening with superlinear growth. Finally, we consider the effect of superlinear growth of the nonlocal energy, \( n > 0 \). To this end, we consider the energy

\[
\text{Min: } F(\gamma) = \int_{-h}^{+h} \left( \tau_0 |\gamma(y)| + \frac{B}{n + 1} \left| \frac{\ell \gamma_y(y)}{\gamma_0} \right|^n \right) dy,
\]

subject to:

\[
\frac{1}{2h} \int_{-h}^{+h} \gamma dy = \bar{\gamma}, \quad \text{and} \quad \gamma(-h) = \gamma(h) = 0.
\]

Enforcing the average deformation constraint (32b) by means of a Lagrange multiplier \( \bar{\tau} \), we are led to the problem

\[
F(\gamma) = \int_{-h}^{+h} \left( \tau_0 |\gamma(y)| - \bar{\tau} \gamma(y) + \frac{B}{n + 1} \left| \frac{\ell \gamma_y(y)}{\gamma_0} \right|^n \right) dy \to \min!
\]

Consider the representation illustrated in Figure 2(b)

\[
\gamma(y) = \bar{\gamma} \varphi(y), \quad \frac{1}{2h} \int_{-h}^{+h} \varphi dy = 1,
\]

where the function \( \varphi(y) \) represents the normalized deformation profile across the layer. Then, the problem (33) is equivalent to minimizing the energy with respect to \( \bar{\gamma} \) and \( \varphi \) separately at fixed \( \bar{\tau} \). Suppose that \( \varphi^*(y) \geq 0 \) is the optimal energy-minimizing profile. Then (33) reduces to

\[
F(\bar{\gamma}) = 2h \left( \tau_0 |\bar{\gamma}| - \bar{\tau} \bar{\gamma} \right) + \frac{B}{n + 1} \frac{\ell^{n+1}}{\gamma_0^{n+1}} C \bar{\gamma} |\bar{\gamma}|^{n+1} \to \min!
\]
with

\begin{equation}
C = \int_{-1}^{+1} |\varphi^*(\xi)|^{n+1} \, d\xi.
\end{equation}

Because of the superlinear growth of the nonlocal energy term in (35), it follows that \( \dot{\gamma} = 0 \) if \( \bar{\tau} < \tau_0 \), and that non-zero average deformations, \( \bar{\gamma} \neq 0 \), can be attained only if

\begin{equation}
|\bar{\tau}| > \tau_0.
\end{equation}

It thus follows that superlinear strain-gradient energies leave the initial yield point unchanged and that, correspondingly, the solids do not exhibit a size effect.

3.4. Discussion. The scaling law (26) greatly overpredicts the observed dependence of the effective shear yield stress on layer thickness for polycrystalline copper, which is of the form

\begin{equation}
\frac{\tau_\ell}{\tau_0} \sim \left( \frac{\ell}{h} \right)^\alpha,
\end{equation}

with \( \alpha \sim 0.2, [2, 13] \). The scaling discrepancy is not removed by strain hardening exponent or superlinear growth of the nonlocal energy. This suggests that the scaling (26) is indeed a direct consequence of the ‘differential structure’ of conventional strain-gradient plasticity, i.e., of the assumption that the nonlocal energy is a function of the first-order derivatives of strain. The discrepancy between (26) and experimental measurements has been pointed out as an intrinsic limitation of conventional strain-gradient plasticity and as a pressing theoretical challenge.

4. Fractional strain-gradient plasticity

We proceed to show that the scaling discrepancy of conventional strain-gradient plasticity can be removed by extending the theory to fractional derivatives of plastic strain, or ‘fractional strain-gradient plasticity’. Specifically, we assume integrated fractional strain-gradient free energies of the form

\begin{equation}
\Psi_g(\varepsilon^p) = \int \int_{\Omega} B \frac{\ell^{s(n+1)}}{\varepsilon_0^{n+1}} \frac{|\varepsilon^p(\mathbf{x}') - \varepsilon^p(\mathbf{x}'')|^{n+1}}{|\mathbf{x}' - \mathbf{x}''|^{d+s(n+1)}} \, dV' \, dV'',
\end{equation}

where \( d \) is the spatial dimension, \( s \in (0, 1) \) is the fractional order of differentiation and we write

\begin{equation}
|\varepsilon| \equiv (\varepsilon_{ij}\varepsilon_{ij})^{1/2}.
\end{equation}

The functional (39) is known as the Gagliardo seminorm [37] and interpolates between the integrated forms of (8) and (9). In particular, (39) is equivalent to integrated form of (8) when \( s \to 0 \) and to the integrated form of (9) when \( s \to 1 \). It is also known [37] that the fractional free energy of order \( s' \) dominates the fractional free energy of order \( s \) if \( 0 < s \leq s' < 1 \). Thus, for \( s \in (0, 1) \) the fractional free energy is ‘softer’ than the classical
gradient free energy (9). We also note that the fractional strain-gradient free energy (39) is now non-local in the sense of entailing a double integral over the domain, describing interactions at a distance, and not just higher-order derivatives of strain.

As an illustrative example of the behavior of the energy (39), consider an infinite solid containing a layer of thickness $2h$ deformed in simple shear and otherwise undeformed. Then, a simple calculation gives (39) as

$$
\Psi_g(\gamma) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{B}{n+1} \frac{\varepsilon^{(n+1)}}{\gamma_0^{n+1}} \left| y' - \gamma(y'') \right|^{n+1} dy' dy'',
$$

with some redefinition of the modulus $B$. Suppose, specifically, that $\gamma(y)$ is constant within the layer. Then, (41) evaluates to

$$
\Psi_g(\gamma) = \frac{4B}{n+1} \frac{\gamma^{n+1}(2h)^{1-s(n+1)}}{\gamma_0^{n+1} s(n+1)(1-s(n+1))},
$$

which is finite for

$$
s > 0, \quad s(n+1) < 1.
$$

In the particular case of linear growth, $n = 0$, (42) further reduces to

$$
\Psi_g(\gamma) = \frac{4B}{\gamma_0} \frac{\gamma^{n+1}(2h)^{1-s}}{s(1-s)}.
$$

We note that the nonlocal energy (22) of strain-gradient plasticity is recovered in the limit of $s \to 1$, provided that the constants are renormalized so as to remain bounded.

4.1. Simple shear of plastically constrained strip revisited. We revisit the strip problem of Section 3. For given $\bar{\gamma} \geq 0$ and assuming local dissipation, the total energy reduces to

$$
\text{Minimize: } F(\gamma) = \int_{-h}^{+h} \tau_0 |\gamma| + \frac{A}{m+1} \frac{\gamma}{\gamma_0^{m+1}} dy
$$

$$
+ \int_{-h}^{+h} \int_{-h}^{+h} \frac{B}{n+1} \frac{\varepsilon^{(n+1)}}{\gamma_0^{n+1}} \left| y' - \gamma(y'') \right|^{n+1} dy' dy'',
$$

subject to: $\frac{1}{2h} \int_{-h}^{+h} \gamma dy = \bar{\gamma}$, $\gamma(-h) = \gamma(0) = 0$,

with some redefinition of the modulus $B$. For simplicity, we specifically consider the case of ideal hardening, $A = 0$ and $m = 0$, and linear growth of the nonlocal energy, $n = 0$. In this particular case, (45) further reduces to

$$
\text{Min: } F(\gamma) = \int_{-h}^{+h} \tau_0 |\gamma| dy + \int_{-h}^{+h} \int_{-h}^{+h} \frac{B}{\gamma_0} \frac{\varepsilon^{(n+1)}}{\gamma(y'')^{n+1}} |y' - y''|^{1+s(n+1)} dy' dy'',
$$

subject to: $\frac{1}{2h} \int_{-h}^{+h} \gamma dy = \bar{\gamma}$, $\gamma(-h) = \gamma(0) = 0$. 


Proceeding as in Section 3.3, we separate $\gamma(y)$ into its mean value $\bar{\gamma}$ and its profile $\varphi(y)$, eqs. (34). Optimizing the profile, the minimum problem further reduces to

$$F = 2h \tau_0 |\gamma| + 2h_0 \ell^s h^{1-s} |\bar{\gamma}| - 2h \bar{\tau} \gamma \rightarrow \min!$$

with some redefinition of the constants. From the linear growth of all the energy terms it follows that $\bar{\gamma} = 0$ necessarily if

$$|\bar{\tau}| < \frac{h + \ell^s h^{1-s}}{h} \tau_0,$$

Conversely, for non-zero minimizers to be possible, we must have

$$|\bar{\tau}| = \frac{h + \ell^s h^{1-s}}{h} \tau_0,$$

which identifies the effective shear yield stress as

$$\tau_\ell = \tau_0 \left[1 + \left(\frac{\ell}{h}\right)^s\right].$$

For thin layers, $h \ll \ell$, (50) reduces to

$$\frac{\tau_\ell}{\tau_0} \sim \left(\frac{\ell}{h}\right)^s.$$

Thus fractional strain-gradient plasticity predicts that the effective shear yield stress scales in inverse proportion to the thickness of the layer raised to the order $s$ of differentiation in the nonlocal energy.

4.2. Discussion. The relaxed scaling law (51) can be made to match the observed scaling law (38) simply by taking

$$s = \alpha,$$

i.e., by identifying the order of differentiation with the observed scaling exponent. As already noted, $\alpha \sim 0.2$ for polycrystalline Cu which, according to the identification (52), suggests that polycrystalline Cu is strongly ‘fractional’, i.e., the order of differentiation in the nonlinear energy is much smaller than 1. Over such range of parameters, fractional strain-gradient plasticity represents a large quantitative and qualitative correction with respect to conventional strain-gradient plasticity.

5. Summary and concluding remarks

Prompted by the mounting experimental evidence pointing to a gaping discrepancy between the size-dependent yield strength of metals and predictions from conventional strain-gradient plasticity, e.g. [17, 2], we have developed a new fractional strain-gradient theory of plasticity (FSGP) that uses fractional derivatives of plastic strain as a means of quantifying the inhomogeneity of plastic deformation. We motivate the theory by means of an analysis of a plastically-constrained shear layer based on conventional
strain-gradient plasticity. The analysis invariably predicts a size scaling exponent $\alpha = 1$ that grossly overestimates the experimentally-observed values. We take this discrepancy to suggest that the differential structure of conventional strain-gradient plasticity is overly stiff and proceed to relax the excessive rigidity by recourse to fractional plastic-strain gradients. Specifically, by allowing the free energy to depend on a fractional derivatives of strain, we show that the size-scaling discrepancy between conventional strain-gradient plasticity and the experimental data is resolved. When applied in the shear layer configuration, the theory predicts a size scaling relation with exponent $\alpha$ equal to the fractional order of plastic strain-gradient differentiation. Through this identification, the observed experimental scaling can be exactly matched by an appropriate choice of fractional differential order.

The form of the non-local fractional plastic strain-gradient contribution to the free energy is explicitly given by a double-integral representation and its fractional differential character is set simply by an appropriate choice of exponents in the interaction kernel. By virtue of this explicit representation, a numerical discretization of the free energy, e.g., by means of finite elements, is rendered straightforward. It should also be straightforward to formally extend the theory to finite deformations through the use of appropriate kinematic and stress variables. In this setting, Conti and Ortiz [33] have shown that fractional strain-gradient theories facilitate topological transitions, such as void coalescence in ductile fracture, which are otherwise hindered by conventional strain-gradient theories. These and other possible extensions and applications of the theory suggest themselves as worthwhile directions for further research.

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